arXiv:1203.3256v2 [cs.CG] 26 Mar 2012

Conflict-free graph orientations with parity constraints *

Sarah Cannon [†‡] Mashhood Ishaque[‡] Csaba D. Tóth [‡§]

Abstract

It is known that every multigraph with an even number of edges has an even orientation (*i.e.*, all indegrees are even). We study parity constrained graph orientations under additional constraints. We consider two types of constraints for a multigraph $G = (V, E)$: (1) an *exact conflict* constraint is an edge set $C \subseteq E$ and a vertex $v \in V$ such that C should not equal the set of incoming edges at v; (2) a *subset conflict* constraint is an edge set $C \subseteq E$ and a vertex $v \in V$ such that C should not be a subset of incoming edges at v. We show that it is NP-complete to decide whether G has an even orientation with exact or subset conflicts, for all conflict sets of size two or higher. We present efficient algorithms for computing parity constrained orientations with *disjoint* exact or subset conflict pairs.

1 Introduction

An *orientation* of an undirected multigraph is an assignment of a direction to each edge. It is well known [14] that a connected multigraph has an even orientation (*i.e.*, all indegrees are even) iff the total number of edges is even. In the *parity constrained orientation* (PCO) problem, we are given a multigraph $G = (V, E)$ and a function $p : V_0 \to \{0, 1\}$ for some subset $V_0 \subseteq V$, and we wish to find an orientation of G such that the indegree of every vertex $v \in V_0$ is $p(v)$ modulo 2, or report that no such orientation exists. This problem has a simple solution in $O(|V| + |E|)$ time [14]. Motivated by applications in geometric graph theory, we consider PCO under additional constraints of the following two types:

1. an *exact conflict* constraint is a pair $(C, v) \in 2^E \times V$ of a set C of edges and a vertex v such that C should not *equal* to the set of incoming edges at v;
2. a *subset conflict* constraint is a pair $(C, v) \in 2^E \times V$ of a set C of edges and a vertex v such that C should not be a *subset* of incoming edges at v;

We denote by PCO-EC and PCO-SC, respectively, the PCO problem with exact conflicts and subset conflicts. We wish to find an orientation of G such that the indegree of every vertex $v \in V_0$ is $p(v)$ modulo 2, and satisfies *all* additional constraints.

*Research was supported by NSF Grants #CCF-0830734 and #CBET-0941538. Work by Tóth was also supported by NSERC grant RGPIN 35586.

†Department of Mathematics, Tufts University, Medford, MA 02155, USA, scanno01@cs.tufts.edu

‡Department of Computer Science, Tufts University, Medford, MA 02155, USA, mishaque@cs.tufts.edu

§Department of Mathematics and Statistics, University of Calgary, AB, Canada, cdtoth@ucalgary.ca

Two (exact or subset) conflicts, (C_1, v_1) and (C_2, v_2), are *disjoint* if $v_1 \neq v_2$ or $C_1 \cap C_2 = \emptyset$. This means that disjoint conflicts at any fixed vertex correspond to disjoint edge sets. Let PCO-DEC (resp., PCO-DSC) denote PCO with pairwise disjoint exact (resp., subset) conflicts. If $|C| = k$ for some integer $k \in \mathbb{N}$ in all conflicts $(C, v) \in 2^E \times V$, we talk about the problems PCO-kEC, PCO-kSC, PCO-kDEC, and PCO-kDSC. If $|C| = 2$ for a conflict (C, v), we say that C is a *conflict pair* of edges.

Results. We show that PCO-EC and PCO-SC are NP-complete, and in fact already PCO-2EC and PCO-2SC are NP-complete. These problems are fixed parameter tractable: if G has m edges and there are s_k conflicts of size $k = 2, 3, \ldots$, then they can be solved in $O((m^{1.5} + n(n+m)) \prod_{k\geq 2}(k+1)^{s_k})$ and $O((n+m) \prod_{k\geq 2} k^{s_k})$ time, respectively. On the other hand, we present polynomial time algorithms for the variants with pairwise disjoint conflicts. Specifically, if the multigraph $G = (V, E)$ has n vertices and m edges, then both PCO-DEC and PCO-DSC can be solved in $O(m^{2.5})$ time. For PCO-2DEC, if no feasible orientation exists, we can compute an orientation with the *maximum* number of vertices satisfying the parity constraints within the same runtimes.

Motivation. An even orientation subject to disjoint exact conflict pairs was a crucial tool in the recent solution of the *disjoint compatible matching conjecture* [2, 12] (see details below). The *exact conflict* constraint differs substantially from typical constraints in combinatorial optimization—it cannot be expressed as a linear inequality with 0-1 variables corresponding to the orientation of the edges. This led us to start a systematic study of PCO-EC. For comparison, we also considered *subset* conflicts, which have a natural integer programming representation. The two types of conflict constraints are indeed quite different.

Application. The exact conflict pair constraint originates from the disjoint compatible matching problem in geometric graph theory. It is clear that every 1-factor (*i.e.*, matching) can be augmented to a 2-factor by adding new edges. This, however, is not always possible if the input is a crossing-free straight-line graph in the plane, and it has to be augmented with compatible (*i.e.*, noncrossing) straight-line edges.

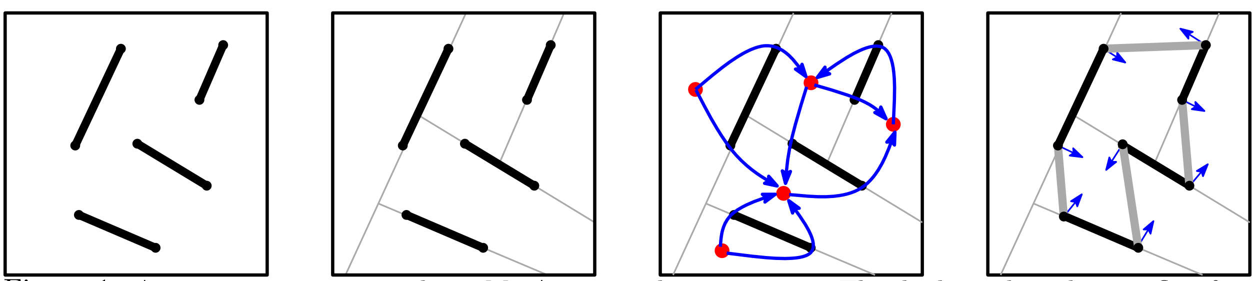

Figure 1: An even geometric matching M. A convex decomposition. The dual graph with a conflict-free even orientation. An augmentation of M to a 2-factor.

It was conjectured [2] that every geometric matching with an *even* number of edges can be augmented to a crossing-free 2-factor (Disjoint Compatible Matching Conjecture). This conjecture has recently been proved [12]. The new edges are added inside the faces of a convex decomposition of the input matching. A crucial lemma in the proof claims that an augmentation to a 2-factor exists iff the *dual graph* of the convex decomposition has an even orientation that avoids a collection of pairwise disjoint exact conflict pairs [2, 12]. Our algorithm for PCO-2DEC can decide whether such an orientation exists in $O(|M|^{2.5})$ time.

Related previous work. Graph orientations are fundamental in combinatorial optimization. It is a primitive often used for representing a variety of other problems. For example, *unique sink*

orientations of polytopes are used for modeling pivot rules in linear programming [15], and *Pfaffian* orientations are used for counting perfect matchings in a graph [14].

Hakimi [11] gave equivalent combinatorial conditions for the existence of an edge orientation with prescribed indegrees. These were generalized by Frank [6] for indegrees of subsets of vertices. Felsner *et al.* [4, 5] computed the asymptotic number of orientations with given indegrees. The graph orientation problem where the indegree of each vertex must be between given upper and lower bounds was solved by Frank and Gyárfás [7]. Frank *et al.* [10] also solved the variant of this problem under parity constraints at the vertices. Frank *et al.* [8] characterized parity constrained graph orientations where the resulting digraph has k edge-disjoint spanning arborescences with given roots. Frank and Király [9] characterized graphs that admit k-edge-connected parity constrained orientations under any parity constraint where sum of parities equals the number of edges modulo 2. Khanna *et al.* [13] devised approximation algorithms for an orientation constrained network design problem, but they do not consider parity or conflict constraints.

Proof techniques and organization. The NP-hardness proofs and our algorithms are broken into elementary reduction steps, each of which uses some simple gadget, that is, a small graph with some carefully placed conflicts. These gadgets are quite remarkable and fun to work with, as they allow for a systematic treatment of all variants of the conflict-free graph orientation problem.

In Section 2, we reduce (1-IN-3)-SAT to PCO-EC and PCO-SC, independently. In Section 3, we first reduce PCO-2DEC to a maximum matching problem in a modified line graph. Then we reduce PCO-DEC, with disjoint conflicts of size *at least 2,* to disjoint conflict *pairs.* Finally, the problem PCO-DSC, with disjoint subset conflicts, is reduced to PCO-2DEC.

2 NP-completeness for exact and subset conflicts

We reduce (1-IN-3)-SAT to each of PCO-2EC and PCO-2SC. It follows that PCO-kEC, PCO-kSC, PCO-EC and PCO-SC are also NP-hard. (1-IN-3)-SAT is known to be an NP-hard problem [3]. It asks whether a boolean expression in conjunctive normal form with 3 literals per clause can be satisfied such that each clause contains exactly one true literal.

2.1 NP-completeness of PCO-2EC

Let I be an instance of (1-IN-3)-SAT with variables $X_1, \ldots, X_n$ and clauses $C_1, \ldots, C_m$. We construct a multigraph $G_I = (V, E)$ and a set $\mathcal{C}_I \subset \binom{E}{2} \times V$ of exact conflict pairs (refer to Fig. 2; arcs denote exact conflict pairs). For each variable X_i, we construct a caterpillar graph as follows. Create a chain of vertices labeled $x_{i1}, x_{i2}, x_{i3}, \ldots, x_{i(2m)}$. Attach three edges to the first and the last vertex of this chain, and attach two edges to all interior vertices of the chain. We call these the *legs* of the caterpillar. At each vertex x_{ij}, let every pair of adjacent edges be an exact conflict pair.

For each clause C_j, create a node c_j. We attach to c_j a leg from each of the three caterpillars corresponding to the variables that appear in C_j. Specifically, if variable X_i appears in clause C_j, attach some edge leaving vertex $x_{i(2j-1)}$ to c_j; if $\overline{X}_i$ appears in clause C_j, attach some edge leaving $x_{i(2j)}$ to c_j. At this point, each node c_j has degree exactly 3, because each clause contains exactly three variables in instance I of (1-IN-3)-SAT. Additionally, for each node c_j create two more nodes a_j and b_j, each connected to c_j by a single edge; make these edges an exact conflict pair. Finally, create an additional node v_0, and connect all "unused" legs of the caterpillars to v_0. If $|E(G_I)|$

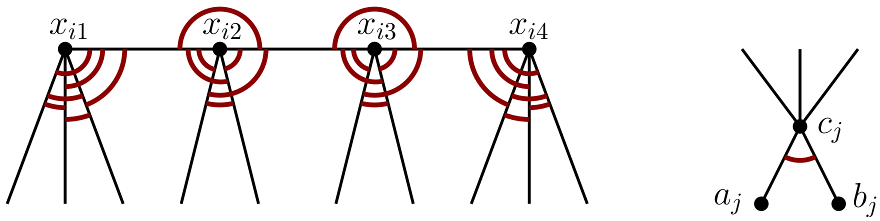

Figure 2: Left: variable gadget. Right: clause gadget.

is odd, create an additional vertex v_0' connected to v_0 by a single edge. We will now show that a conflict-free even orientation of G_I corresponds to a true instance I of (1-IN-3)-SAT.

Lemma 1. *Instance I of* (1-IN-3)-SAT *is true iff there is an exact conflict-free even orientation for graph G_I and set $\mathcal{C}_I$ of exact conflict pairs.*

Proof. Assume instance I of 1-IN-3-SAT is true. That is, there is a valid truth assignment for all variables X_i such that each clause C_j contains exactly one true literal. We construct a conflict-free even orientation for G_I and $\mathcal{C}_I$ as follows. If variable X_i is true, then for all ℓ, orient all edges incident to $x_{i(2\ell-1)}$ towards $x_{i(2\ell-1)}$, and all edges incident to $x_{i(2\ell)}$ away from $x_{i(2\ell)}$. Note that the indegree of the vertices $x_{i\ell}$ in the caterpillar are alternately 0 and 4, and hence no exact conflict pair equals the set of edges oriented into one of these nodes. Similarly, if variable X_i is false, then orient all edges incident to $x_{i(2\ell-1)}$ away from $x_{i(2\ell-1)}$, and all edges incident to $x_{i(2\ell)}$ towards $x_{i(2\ell)}$. Orient each edge a_jc_j and b_jc_j towards c_j. The legs of caterpillars oriented away from c_j correspond to a 'true' assignment while legs oriented into c_j correspond to a 'false' assignment. Since exactly one literal in each C_j is true, exactly one of 5 incident edges is oriented away from c_j. That is, the indegree of each c_j is 4. We now have a conflict-free even orientation for G_I and $\mathcal{C}_I$, as required.

Assume now that there is a conflict-free even orientation for G_I and $\mathcal{C}_I$. The parity constraints and the conflict pairs ensure that all 4 edges incident to each $x_{i\ell}$ are oriented either to $x_{i\ell}$ or away from $x_{i\ell}$. Therefore, the indegrees of the nodes $x_{i1}, \ldots, x_{i(2m)}$ are alternately 0 and 4. If all incident edges are oriented into x_{i1}, then set X_i 'true,' otherwise set X_i 'false.' Our construction ensures that the indegree of each c_j is exactly four. Since both a_jc_j and b_jc_j must be oriented into c_j, exactly two legs of some caterpillars are oriented to c_j (and exactly one away from c_j). This guarantees that each C_j contains exactly two false literals and one true literal, and so this truth assignment for all variables is a valid solution to instance I of 1-IN-3-SAT. □

By augmenting the conflict sets by additional edges, if necessary, we see that PCO-kEC is also NP-hard. It is clear that these problems are in NP: one can check in linear time whether the parity and all additional constraints are satisfied.

Theorem 1. *Problems* PCO-EC *and* PCO-kEC*, for every $k \geq 2$, are NP-complete.*

2.2 NP-completeness of PCO-2SC

We now reduce (1-IN-3)-SAT to PCO-2SC. Let I be an instance of (1-IN-3)-SAT with variables $X_1, \ldots, X_n$ and clauses $C_1, \ldots, C_m$. We construct a multigraph $G_I = (V, E)$ and a set $\mathcal{C}_I \subset \binom{E}{2} \times V$ of subset conflict pairs (Fig. 3; arcs denote subset conflict pairs). For each variable X_i, create a circuit $(x_{i1}, x_{i2}, \ldots, x_{im})$ of length m. Label the edge connecting $x_{i\ell}$ and $x_{i,\ell+1}$ as z_{il}. (All arithmetic

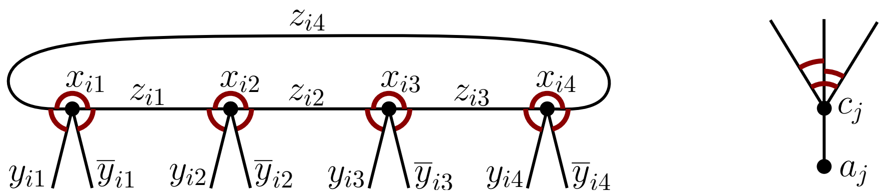

Figure 3: Left: variable gadget. Right: clause gadget.

with index ℓ is performed modulo m). To each $x_{i\ell}$, attach two additional edges, $y_{i\ell}$ and $\overline{y}_{i\ell}$. Mark the edge pairs $\{z_{i\ell}, z_{i,\ell+1}\}$, $\{z_{i\ell}, y_{i,\ell+1}\}$, and $\{z_{i\ell}, \overline{y}_{il}\}$ as subset conflict pairs.

For each clause C_j, create a node c_j. If X_i is in clause C_j, attach the edge y_{ij} to c_j. Similarly, if $\overline{X_i}$ occurs in clause C_j, attach the edge $\overline{y}_{ij}$ to c_j. At this point, each node c_j should have degree 3, since each clause in instance I of (1-IN-3)-SAT contains three variables. Label every pair of edges incident on c_j as a subset conflict pair. Additionally, for each node c_j create one more node a_j connected to c_j by a single edge. Finally, create an additional node v_0 and connect it to all "unused" edges $y_{i\ell}$ or $\overline{y}_{i\ell}$; if $|E(G_I)|$ is odd, create node v_0' and connect it to node v_0 by a single edge.

Lemma 2. *Instance I of* (1-IN-3)-SAT *is true iff there is a subset conflict-free even orientation for graph G_I and set $\mathcal{C}_I$ of subset conflict pairs.*

Proof. Assume that instance I of (1-IN-3)-SAT is true. That is, there exists a truth assignment for all variables X_i such that exactly one literal in each clause C_j is true. If X_i is true, orient edge $z_{i\ell}$ from $x_{i\ell}$ to $x_{i,\ell+1}$ for all ℓ; orient $y_{i\ell}$ away from $x_{i\ell}$; and orient $\overline{y}_{i\ell}$ into $x_{i\ell}$. Then, at each $x_{i\ell}$, the indegree is 2, but no two conflicting edges are oriented into $x_{i\ell}$. If X_i is false, orient all edges of the variable gadget in the opposite way. Since each C_j has exactly one true literal, exactly one of the three edges from variable gadgets is oriented into c_j. Orient edge a_jc_j into c_j; now, the indegree of both a_j and c_j is even, and no two edges oriented into c_j are in conflict. We have constructed a conflict-free even orientation of G_I.

Assume now that there exists a conflict-free even orientation for G_I and $\mathcal{C}_I$. The subset conflict pairs along the circuit $(x_{i1}, \ldots, x_{im})$ ensure that the circuit is oriented cyclically. If z_{i1} is oriented away from x_{i1}, then set X_i to 'true,' otherwise to 'false.' In either case, exactly one edge of the circuit is oriented into each $x_{i\ell}$. Since the indegree has to be even, exactly one of $y_{i\ell}$ and $\overline{y}_{i\ell}$ is oriented into $x_{i\ell}$. The subset conflicts imply that if X_i is true, $\overline{y}_{i\ell}$ is oriented into $x_{i\ell}$ and $y_{i\ell}$ is oriented away, while if X_i is false, $y_{i\ell}$ is oriented into $x_{i\ell}$ and $\overline{y}_{i\ell}$ is oriented away. In other words, edges oriented towards c_j correspond to an assignment of 'true' for the corresponding variable, while edges oriented away from c_j correspond to assignments of 'false' for the corresponding variables. For each node c_j, the conflicts we imposed ensure that exactly two edges are oriented into c_j, and one of them is a_jc_j. Hence, exactly one variable in each clause has been set to true, as required. □

By augmenting the conflict sets with additional edges, if necessary, we see that PCO-kSC is also NP-hard. It is clear that these problems are in NP: one can check in linear time whether the parity and all additional constraints are satisfied.

Theorem 2. *Problems* PCO-SC *and* PCO-kSC*, for every $k \geq 2$, are NP-complete.*

2.3 Fixed Parameter Tractability

We now show that PCO-EC and PCO-SC are fixed-parameter tractable. If G has m edges and there are s_k conflicts of size $k = 2, 3, \ldots$, then these problems can be solved in $O((m^{1.5} + n(n + m)) \prod_{k \geq 2}(k+1)^{s_k})$ and $O((n+m) \prod_{k \geq 2} k^{s_k})$ time, respectively.

First we consider PCO-SC. For each subset conflict set S of size k incident on vertex v, in any valid parity constrained orientation at least one edge of S must be oriented away from v. Arbitrarily choose one edge e from each conflict set S to be oriented away; there are at most $\prod_{k \geq 2} k^{s_k} = O(1)$ ways to do this. Call this set of selected edges E^*. For every edge e in E^*, where e is part of subset conflict set S_e, e connects v_e, the vertex on which all edges in S_e are incident, to w_e, some other vertex. To form a new graph G', remove edge e and if node w_e has a parity constraint, reverse it. A parity constrained graph orientation on G' yields a solution to PCO-SC on G, obtained by reinserting all edges e in E^* with an orientation away from v_e towards w_e. If there is no parity constrained orientation on G', repeat for some other set E^*. If no set E^* yields a parity constrained orientation on G', there is no solution to PCO-SC on G. If m is the number of edges in G, which differs from the number of edges in G' only by a constant, then it is known that each execution of the PCO algorithm takes $O(n+m)$ time [14]. The PCO algorithm is run at most $\prod_{k \geq 2} k^{s_k}$ times, giving a total polynomial runtime of $O((n+m) \prod_{k \geq 2} k^{s_k})$.

Now we consider PCO-EC. We will reduce PCO-EC to PCO-VD, the parity constrained orientation problem in which each vertex has a minimum indegree given by a vertex demand function $F : V \to \mathbb{Z}$; this problem has already been solved by Frank *et al.* [10]. For each exact conflict set S of size k incident on vertex v, in any valid parity constrained orientation either (1) at least one edge must be oriented away, or (2) all edges are oriented toward v and v has indegree more than k. For each exact conflict set, either choose an edge e to orient away from v or place a demand of $k+1$ on node v. There are at most $\prod_{k \geq 2}(k+1)^{s_k} = O(1)$ ways to do this. Call the set of selected edges E^* and the set of selected vertices V^*. Remove the edges in E^* as described previously, flipping the parity requirements of nodes w_e, to form graph G'. Search for a solution to PCO-VD on G'; such a solution yields a solution to PCO-EC when all edges e in E^* are reinserted and oriented away from v_e, towards w_e. Run for all possible sets $E^* \cup V^*$; if no valid solution to PCO-VD on G' is found, there is no valid solution to PCO-EC on G. If n is the number of vertices and m is the number of edges in G, then solving PCO-VDrequires time $O(m^{1.5} + n(n+m))$ and this algorithm runs in polynomial time at most $O((m^{1.5} + n(n+m)) \prod_{k \geq 2}(k+1)^{s_k})$.

3 Polynomial time algorithms

In this section we present polynomial time algorithms for PCO-DEC and PCO-DSC. We start by showing that in most cases we can restrict our attention to *even* orientations. In the *even orientation* problem (EO), we are given a multigraph $G = (V, E)$, and we wish to find an orientation of G where *every* vertex has even indegree. Analogously to the variants of PCO with additional constraints, we introduce the problems EO-EC, EO-DEC, and EO-kDEC for exact conflicts and EO-SC, EO-DSC and EO-kDSC for subset conflicts.

We reduce PCO and most of its variants to the corresponding even orientation problems. The notable exception is PCO-DEC. We prove that the reduction holds for the more general optimization version of these problems as well. In the optimization version of PCO with possible additional constraints, we wish to find a conflict-free orientation which satisfies the maximum number of parity constraints.

Lemma 3. *The optimization versions of* PCO, PCO-EC*,* PCO-SC*, and* PCO-DSC*, can be reduced to the corresponding version of* EO *in linear time.*

Proof. We first reduce the parity constrained orientation (PCO) problem to the even orientation problem (EO), and then consider various additional constraints. Consider an instance I_1 of PCO, that is, a multigraph $G_1 = (V_1, E_1)$ with a partial parity constraint $p_1 : V_0 \to \{0, 1\}$, $V_0 \subseteq V_1$. We construct an instance I_3 of EO in two steps.

Step 1: We construct an instance I_2 of PCO by augmenting G_1 to a multigraph $G_2 = (V_2, E_2)$ with new edges and vertices such that all parity constraints are even. For *each* vertex $v \in V_0$ with odd constraint $p_1(v) = 1$, add a new (dummy) vertex $v' \in V_2$ and a new edge $vv' \in E_2$, with $p_2(v) = 0$ and $p_2(v') = 0$. Let V' be the set of dummy vertices. If G_1 has an orientation satisfying t_1 out of $|V_0|$ parity constraints, then G_2 has an orientation satisfying $t_1 + |V'|$ out of $|V_0| + |V'|$ parity (*i.e.*, evenness) constraints. Indeed, just orienting each dummy edge away from the dummy vertex means every dummy vertex has indegree 0, and the indegree of all adjacent vertices changes parity from odd to even. Conversely, if G_2 has an orientation satisfying the maximum number of parity constraints, say t_2, then we can assume that all dummy edges are oriented away from the dummy vertices. After deleting all dummy edges and vertices, we obtain an orientation of G_1 satisfying $t_2 - |V'|$ parity constraints.

Step 2. Consider an instance I_2 of PCO: a multigraph $G_2 = (V_2, E_2)$ with even parity constraints $p_2 : V_0 \to \{0\}$ for some $V_0 \subseteq V_2$. We construct a new instance of PCO in which the parity of *every* vertex is constrained to be even. Construct $G_3 = (V_3, E_3)$ from G_2 by adding one new (dummy) vertex w, and connecting every vertex $v \in V_2 \setminus V_0$ to w. If $|E_3|$ is odd, add one additional vertex w' connected to w by a single edge. Set the parity constraint of every vertex in V_3 to *even*. If G_2 has an orientation satisfying t_2 out of $|V_0|$ parity constraints, then G_3 has an orientation satisfying $t_2 + |V_2 \setminus V_0| + 1$ parity constraints, just by orienting each dummy edge to make the parity of each unconstrained vertex even. Conversely, if G_3 has an orientation satisfying t_3 parity constraints, then after deleting the dummy vertices and edges (and also removing the parity constraints from vertices in $V_2 \setminus V_0$) we obtain an orientation of G_2 satisfying $t_3 - |V_2 \setminus V_0| - 1$ parity constraints.

In PCO-EC, an instance I includes a family $\mathcal{C}$ of exact conflict constraints. We modify $\mathcal{C}$ as well in two steps. In the first step, we replace every conflict (C, v) where $v \in V_0$, $|C|$ is *odd* and $p_1(v) = 1$, with a new conflict $(C \cup \{vv'\}, v)$. In the second step, we replace every conflict (C, v) where $|C|$ is *odd* and $v \in V_2 \setminus V_0$ with a new conflict $(C \cup \{vw\}, v)$. These modifications ensure that after removing the dummy edges and vertices, the set of incoming edges are not in conflict at any vertex.

In PCO-SC and PCO-DSC, an instance I includes a family $\mathcal{C}$ of subset conflict constraints. When we augment G with new (dummy) vertices and edges, we preserve all these constraints. Independent of the orientation of the dummy edges, the constraints are satisfied in all feasible orientations for I_1, I_2, and I_3. □

Remarks. With the above argument, every instance of PCO-DEC can be reduced to an instance of EO-EC, but the conflicts are no longer disjoint when we augment all conflicts at a vertex v with a common dummy edge.

3.1 Even orientations with disjoint exact conflict pairs

Let $G = (V, E)$ be a connected multigraph, and let $\mathcal{C} \subseteq \binom{E}{2} \times V$ be a family of pairwise disjoint exact conflict pairs. We wish to find an orientation for G with a maximum number of even vertices

such that whenever a vertex v has indegree 2 from edges e_1 and e_2, then $(\{e_1, e_2\}, v) \notin \mathcal{C}$. We present a polynomial time algorithm that either constructs an optimal orientation or reports that none exists. Without loss of generality, assume that G is connected.

Recall the definition of the **line graph** $L(G)$. Given a multigraph $G = (V, E)$, the nodes of $L(G)$ correspond to E, and two nodes are adjacent iff the corresponding edges of G are adjacent. For a multigraph $G = (V, E)$ and conflict pairs $\mathcal{C} \subseteq \binom{E}{2} \times V$, we define the following subgraph of $L(G)$. Let $L' = L'(G, \mathcal{C})$ be the graph whose node set is E, and two nodes $e_1, e_2 \in E$ are adjacent in L' iff they have a common endpoint $v \in V$ *and* $(\{e_1, e_2\}, v) \notin \mathcal{C}$. We show that an instance of the optimization version of EO-2DEC for G and $\mathcal{C}$ reduces to a maximum matching over $L'(G, \mathcal{C})$.

Lemma 4. *Let $G = (V, E)$ be a multigraph with disjoint exact conflict pairs $\mathcal{C}$. There are t vertices with odd indegree in a conflict-free orientation of G that maximizes the number of even vertices iff there are t nodes uncovered in a maximum matching of $L' = L'(G, \mathcal{C})$.*

Proof. First, suppose that a maximum matching M of L' covers all but t nodes. We construct a conflict-free orientation for G. For every edge $(e_1, e_2) \in M$, direct both e_1 and e_2 towards one of their common endpoints in G. We obtain a partial orientation of G, where all indegrees are even, since pairs of edges are directed towards each vertex of G. Since adjacent but conflicting edges are not connected in L', they are not matched in M, and thus there is no vertex in G with indegree 2 where the two incoming edges are in conflict.

Now consider the set of unmatched nodes of L', which is a set $E^* \subseteq E$ of edges in G of size $|E^*| = t$. Out of any three edges incident to a common vertex, at least two can be matched, since the conflict pairs do not overlap. Hence each vertex $v \in V$ is incident to at most two edges in E^*; and if v is incident to two edges in E^*, then those edges are in conflict. So the edges in E^* form disjoint paths and circuits in G. We can orient the edges in E^* into distinct vertices in V. We obtain an orientation of G with exactly t odd vertices.

Next suppose that in a conflict-free orientation of G with the largest number of even vertices, there are exactly t odd vertices. We construct a matching of L'. Consider a vertex $v \in V$. Partition the incoming edges of v into two subsets whose size differ by at most one such that conflicting pairs are in different classes. This is possible, since the conflict pairs are disjoint, and so every edge participates in at most one conflict pair at v. Fix a maximum matching between the two classes arbitrarily. We have matched adjacent edges, but no conflicting pairs. If v is even, then the matching is perfect, otherwise one edge remains unmatched. After repeating this for all vertices $v \in V$, we obtain a matching of L' that covers all but t edges in E. $\square$

We use the following algorithm for constructing a desired even orientation. Given a multigraph G and disjoint exact conflict pairs $\mathcal{C}$, construct graph $L' = L'(G, \mathcal{C})$, compute a maximum matching M on L', and convert it into an orientation of G. For a graph $G = (V, E)$ with n vertices and m edges, the line-graph L' has m nodes and $O(m^2)$ edges. The general max-flow algorithm used to find maximum matchings runs in time cubic in the number of nodes, or in $O(m^3)$ time. Since L' is a unit-capacity graph, Dinic's blocking flow algorithm [1] gives a runtime of $O(m^{2.5})$.

3.2 Even orientations with disjoint exact conflicts

We reduce PCO-DEC with conflicts of size *at least* two to EO-2DEC in linear time. (Recall that PCO-DEC has not been reduced to EO-DEC in Section 3.1). A key ingredient of the reduction is a "switching network" that can rearrange the orientations of k edges of a conflict. This auxiliary network is defined as a graph N_k with parity constraints and disjoint exact conflict pairs. It has $2k$

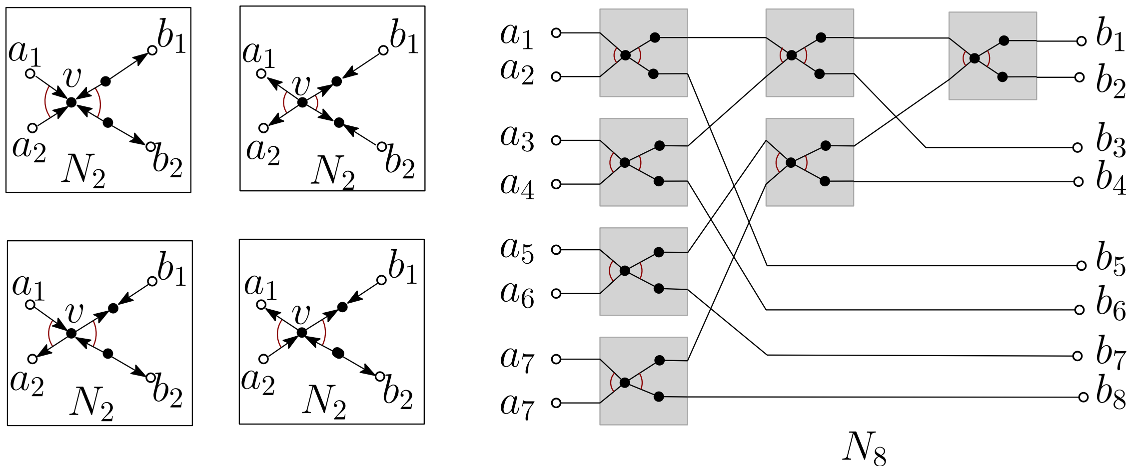

Figure 4: Left: N_2 with four possible orientations at a_1 and a_2. Right: N_8 is composed of 7 copies of N_2.

leaves: k *input* leaves $a_1, \ldots, a_k$ and k *output* leaves $b_1, \ldots, b_k$. We draw N_k in the plane such that the input leaves are on the left side, the output leaves are on the right side, and so it is convenient say that the orientation of each edge is either left-to-right (for short, *right*) or right-to-left (*left*). The network N_k will have the following two properties:

$\mathbf{P}_1$ If exactly ℓ input edges are oriented right, for some $0 \le \ell \le k$, then exactly ℓ output edges are oriented right in *every* valid orientation of N_k.

$\mathbf{P}_2$ If exactly ℓ input edges are oriented right, for some $0 < \ell < k$, then b_1 is oriented right and b_2 is oriented left in *some* valid orientation of N_k.

Properties $\mathbf{P}_1$ and $\mathbf{P_2}$ imply that outputs b_1 and b_2 represent all k inputs for the purposes of exact conflicts. If all inputs are oriented right, then both b_1 and b_2 are oriented right; if no input is oriented right, then neither b_1 nor b_2 is oriented right. If some inputs are oriented right some are left, then there is a valid orientation where b_1 is oriented right and b_2 is left.

For $k = 2$, let N_2 be the graph shown in the left of Fig. 4 (arcs denote *exact* conflict pairs). The leaves may have arbitrary indegrees, but every nonleaf vertex must have even indegree.

For every $k > 2$, the graph N_k is composed of multiple copies of N_2, similarly to a multi-stage switching network where the *switches* correspond to copies of N_2. Specifically, N_k consists of $\lceil \log k \rceil$ stages. Stage $i = 1, \ldots, \lceil \log k \rceil$ consists of $\lceil k/2^i \rceil$ copies of N_2. For each copy of N_2 at stage $i = 1, 2, \ldots \lceil \log k \rceil - 1$, one output leaf is identified with an input leaf in the next stage, and the other output leaf becomes an output leaf of N_k. Refer to the right of Fig. 4 for an example with $k = 8$. Note that graph N_k has at most $6k$ nonleaf vertices.

Lemma 5. *For every $k \in \mathbb{N}$, $k \ge 2$, graph N_k satisfies both $\mathbf{P}_1$ and $\mathbf{P}_2$.*

Proof. For $k = 2$, the two disjoint conflict pairs at v ensure that if both input edges are oriented right, then the indegree of v is 4; if neither input edge is oriented right, then the indegree of v is 0. If exactly one input edge is oriented right, then the indegree of v is 2, and the second incoming edge may be any one of the two edges on the right side of v. It is now easy to verify that properties $\mathbf{P}_1$ and $\mathbf{P}_2$ hold.

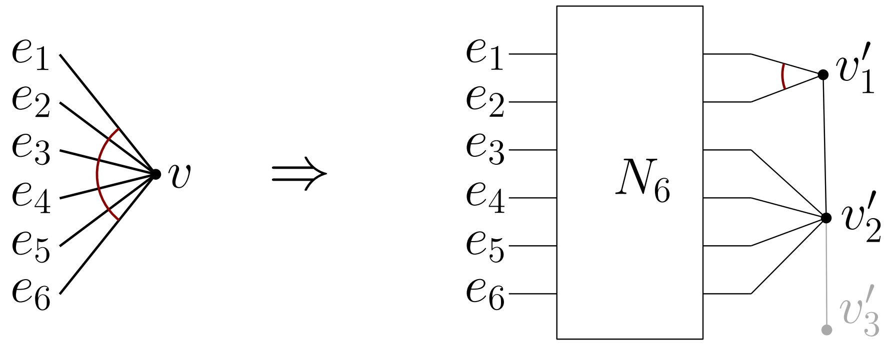

Figure 5: An exact conflict $(\{e_1, \ldots, e_6\}, v)$ is replaced by a network N_6 with the first two outputs identified with v_1 and all remaining outputs identified with v_2.

For $k > 2$, property $\mathbf{P_1}$ follows from the fact that N_2 has this property and we identified input edges with output edges in adjacent copies of N_2. For $\mathbf{P}_2$, assume that not all input edges have the same orientation. Consider an arbitrary valid orientation of N_k. If the two input edges of the rightmost copy of N_2 have different orientations, then $\mathbf{P}_2$ follows. Suppose that these two edges have the same orientation, say, both are oriented right. We show that N_k has another valid orientation where these two edges have different orientations. Let a_i be an input edge of N_k oriented *left*. Note that N_k contains a path from a_i to the rightmost copy of N_2. In all copies of N_2 along this path, there is a valid orientation such that the edges between consecutive copies of N_2 are oriented left. Combining these orientations, we obtain a valid orientation where b_1 is oriented left and b_2 right, as required. $\square$

Let I be an instance of PCO-DEC with conflicts of size *at least* 2. That is, I consists of a multigraph graph $G = (V, E)$, a family of disjoint exact conflicts $\mathcal{C} \subseteq 2^E \times V$ each with at least two edges, and parity constraints $p : V_0 \to \{0, 1\}$. We may assume that at every vertex $v \in V_0$, the number of edges in each conflict set is $p(v)$ modulo 2, since all other conflict constraints are automatically satisfied. We create an instance I' of EO-2DEC, that is, a multigraph $G' = (V', E')$ with disjoint conflict *pairs* $\mathcal{C}' \subseteq \binom{E'}{2} \times V'$ such that G' has a conflict-free even orientation iff G has a valid orientation.

For every vertex $v \in V$, we create a path $\pi'(v)$ in G' as follows: If $p(v) = 0$, then $\pi'(v) = (v'_1, v'_2, v'_3)$ with three nodes; if $p(v) = 1$ or $v \notin V_0$, then $\pi'(v) = (v'_1, v'_2)$ with two nodes. In order to balance the parity of unrestricted nodes $v \in V \setminus V_0$, we create one common auxiliary vertex $u'_0 \in V$, and connect it to v'_2 for every $v \in V \setminus V_0$. If $|V \setminus V_0|$ is odd, we also add a dummy vertex u'_1 and a dummy edge $u'_0u'_1$ ($u'_0u'_1$ is oriented into u'_0 in any even orientation of G').

For each edge $e \in E$, we create an edge $e' \in E'$ as follows. If e is incident to v and it is not part of any conflict at v, then let e' be incident to v_1. For each conflict *pair* $(\{e_1, e_2\}, v)$, let the corresponding edges, e'_1 and e'_2, be incident to v_1, and let $(\{e'_1, e'_2\}, v_1) \in \mathcal{C}'$ be an exact conflict pair. Finally, for each conflict $(\{e_1, \ldots, e_k\}, v) \in \mathcal{C}$, of size $k \geq 3$, we create a copy of the network N_k: identify the edges $e_1, \ldots e_k$ with the input leaves of N_k, identify output leaves b_1 and b_2 with v_1 forming an exact conflict pair at v_1, and identify the remaining $k - 2$ output leaves with v_2. Fig. 5 shows an example for $k = 6$. This completes the specification of the new instance I' of EO-2DEC.

Lemma 6. *Instance I of* PCO-DEC *with G, $\mathcal{C}$, and parity constraints p has a conflict-free orientation iff instance I' of* EO-2DEC *with G' and $\mathcal{C}'$ has a conflict-free even orientation.*

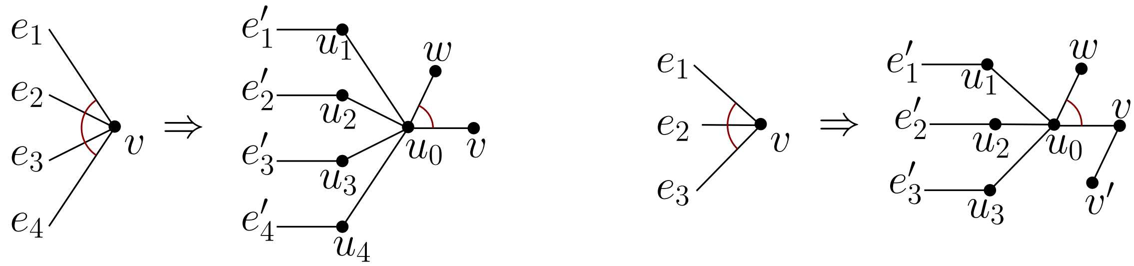

Figure 6: Left: Modification for a subset conflict $(\{e_1, e_2, e_3, e_4\}, v)$ of even size. Right: Modification for a subset conflict $(\{e_1, e_2, e_3\}, v)$ of odd size.

Proof. Assume G has a conflict-free parity constrained orientation o. We construct a conflict-free even orientation o' for G'. Every edge $e \in E$ corresponds to an edge $e' \in E'$. We set the orientation of e' to be the same as e. It remains to specify the orientation of auxiliary structures. For every vertex $v \in V$, we orient edge $v'_1 v'_2 \in E'$ to make the indegree of v'_1 even; and then the possible edges $v'_2 v'_3$ or $v'_2 u'_0$ are oriented to make the indegree of v'_2 even. Since o' satisfies the parity constraints at every vertex $v \in V$, and we added an dummy edge $v'_3 v'_2$ oriented into v_2, it follows that the indegrees of all vertices v'_1, v'_2, and (if exists) v'_3 are even. Next, we choose the orientations in the networks N_k. For a conflict set $(\{e_1, \ldots, e_k\}, v) \in \mathcal{C}$, a network N_k forwards two edges to a conflict pair at v_1 and the remaining $k-2$ edges to v_3. By properties $\mathbf{P}_1$ and $\mathbf{P}_2$, the conflict pair has 0 (resp., 1 or 2) edges oriented into v_1 iff $\{e_1, \ldots, e_k\}$ has 0 (resp., $0 < \ell < k$ or k) edges oriented into v. This implies that if o has no conflict at $v \in V$, then o' has no conflict at $v'_1 \in V'$.

Assume now that G' has a conflict-free even orientation o'. We construct a conflict-free parity-constrained orientation o on G. Recall that every edge $e \in E$ corresponds to an edge $e' \in E'$. Let each e take the same orientation in o as e' has in o'. Suppose that the set of incoming edges at a vertex $v \in V$ equals a conflict set $\{e_1, \ldots, e_k\}$. Then the set of incoming edges of v'_1 is the conflict pair $\{e'_1, e'_2\}$, that is, o' is not a conflict-free orientation. It follows that o is a conflict-free orientation. □

3.3 Even orientations with disjoint subset conflicts

We reduce EO-DSC to EO-2DEC in linear time. Let I be an instance of EO-DSC, that is, a multigraph $G = (V, E)$ with disjoint subset conflicts $\mathcal{C}$ of various sizes. We construct a new multigraph $G' = (V', E')$ with disjoint exact conflict *pairs* such that G has a conflict-free even orientation iff G' does.

The graph G' is constructed by modifying $G = (V, E)$. We make some local modifications for each subset conflict $(\{e_1, \ldots, e_k\}, v) \in \mathcal{C}$. If k is even, then replace the edges $e_1, \ldots, e_k$ with the configuration shown in Fig. 6(left) with $k+2$ new vertices $u_0, u_1, \ldots, u_k, w$ and one new exact conflict pair $(\{u_0 v, u_0 w\}, u_0) \in \mathcal{C}'$. If k is even, then replace the edges $e_1, \ldots, e_k$ with the configuration shown in Fig. 6(right) with $k+3$ new vertices $u_0, u_1, \ldots, u_k, v', w$ and one new exact conflict pair $(\{u_0 v, u_0 w\}, u_0) \in \mathcal{C}'$. By construction, the new exact conflict pairs in $\mathcal{C}'$ are pairwise disjoint.

Lemma 7. *Instance I of* EO-DSC *with G and subset conflict $\mathcal{C}$ has a conflict-free even orientation iff instance I' of* EO-2DEC *with G' and $\mathcal{C}'$ has a conflict-free even orientation.*

Proof. Assume G has a conflict-free even orientation o. We construct a conflict-free even orientation o' for G'. The common edges of G and G' should have the same orientation as in o. For each conflict

$(\{e_1, \ldots, e_k\}, v) \in \mathcal{C}$, let the orientation of the edges $e'_1, \ldots, e'_k \in E'$ be the same as $e_1, \ldots, e_k \in E$, respectively. Since o is conflict-free, some edge $e_i \in E$, $1 \leq i \leq k$, is oriented away from v. The corresponding edge $e'_i \in E'$ is oriented away from u_i, and so u_0u_i is oriented into u_0. Therefore, $\{u_0v, u_0w\}$ cannot be the set of edges oriented into u_0. Since the indegrees of $u_0, u_1, \ldots, u_k$ are even (0 or 2), they uniquely determine the orientation of *all* edges incident to u_0. Note also that the edge wu_0 is always oriented into v because the indegree of w must be 0. It follows that u_0v is oriented into v in o' iff an odd number of edges in $\{e_1, \ldots, e_k\}$ are oriented into v in o. That is, the contribution of a conflict to the parity of v is the same as the contribution of the edge u_0v in o'. Overall, if o is an even orientation, then o' is even as well.

Assume now that G' has a conflict-free even orientation o'. We construct a conflict-free even orientation o for G. The common edges of G and G' should have the same orientation as in o'. For each conflict $(\{e_1, \ldots, e_k\}, v) \in \mathcal{C}$, let the orientation of the edges $e_1, \ldots, e_k \in E$ be the same as $e'_1, \ldots, e'_k \in E'$, respectively. Since o' is conflict-free and the indegree of u_0 is even, some edge u_0u_i, $1 \leq i \leq k$, is oriented into u_0. This means that $e'_i \in E'$ is oriented away from u_i, and the corresponding edge $e_i \in E$ is not oriented into v. Hence, not all edges in $\{e_1, \ldots, e_k\}$ are orieted into v. All vertices of G preserve their parity, hence o is even as well. $\square$

4 Conclusion

We have shown that the parity constrained orientation problem is NP-hard in the presence of exact or subset conflicts, and in fact already in the presence of conflict pairs. On the other hand, the problems are in P for disjoint conflict pairs. It remains an open problem to determine the status of PCO-DEC if all conflicts have one or two edges; while subset conflict sets with one edge are trivial, exact conflict sets with one edge are not, and our reductions only apply to exact conflicts with two or more edges.

References

[1] R. K. Ahuja, T. L. Magnanti, and J. B. Orlin, *Network Flows: Theory, Algorithms, and Applications*, Prentice Hall, February 1993.

[2] O. Aichholzer, S. Bereg, A. Dumitrescu, A. García, C. Huemer, F. Hurtado, M. Kano, A. Márquez, D. Rappaport, S. Smorodinsky, D. Souvaine, J. Urrutia, and D. R. Wood, Compatible geometric matchings, *Comput. Geom.* **42** (2009), 617-626.

[3] S. Arora and B. Barak, *Computational Complexity: A Modern Approach*, Cambridge Univ. Press, 2009.

[4] S. Felsner, É. Fusy, and M. Noy, Asymptotic enumeration of orientations, *Discrete Math. Theor. Comp. Sci.* **12** (2) (2010), 249–262.

[5] S. Felsner and F, Zickfeld, On the number of planar orientations with prescribed degrees, *Electron. J. Comb.* **15** (1) (2008), article R77.

[6] A. Frank, On the orientaiton of graphs, *J. Combin. Theor. B* **28** (1980), 251–261.

[7] A. Frank and A. Gyárfás, How to orient the edges of a graph, *Coll. Math. Soc. J. Bolyai* **18** (1976), 353-364.

[8] A. Frank, T. Jordán, and Z. Szigeti, An orientation theorem with parity conditions, *Discrete Appl. Math.* **115** (2001), 37–47.

[9] A. Frank and Z. Király, Graph orientations with edge-connection and parity constraints, *Combinatorica* **22** (2002), 47-70.

[10] A. Frank, É. Tardos, and A. Sebő, Covering directed and odd cuts, *Math Prog. Stud.* **22** (1984), 99–112.

[11] S. L. Hakimi, On the degrees of the vertices of a directed graph, *J. Franklin Inst.* **279** (1965), 280–308.

[12] M. Ishaque, D. L. Souvaine, and C. D. Tóth, Disjoint compatible geometric matchings, in *Proc. 27th Sympos. on Comput. Geom.*, ACM, 2011, pp. 125–134.

[13] S. Khanna, J. Naor, and F. B. Shepherd, Directed network design with orientation constraints, *SIAM J. Discre. Math.* **19** (2005), 245–257.

[14] L. Lovász and M. D. Plummer, *Matching Theory*, AMS Chelsea, 2009.

[15] T. Szabó and E. Welzl, Unique sink orientations of cubes, in *Porc. 42nd FOCS*, IEEE, 2001, pp. 547–555.